

Derivation of Invariant Varieties of Periodic Points from Singularity Confinement in the case of Toda Map

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In our previous work we have shown that the invariant varieties of periodic points (IVPP) of all periods of the 3 dimensional Lotka-Volterra map can be derived, iteratively, from the singularity confinement (SC). The method developed there can be applied to any integrable maps of dimension d only when the number of the invariants p equals to $d - 1$. We propose, in this note, a new algorithm of the derivation which can be used in the cases $\frac{d}{2} \leq p \leq d - 2$. Applying this algorithm to the 3 point Toda map, we derive a series of its IVPP's.

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I. INTRODUCTION

Let us consider a map

$$F : x = (x_1, x_2, \dots, x_d) \rightarrow X = (X_1, X_2, \dots, X_d), \quad x, X \in P^d\mathbb{C} \quad (1)$$

in d dimensional projective complex space. Since we consider rational maps it is convenient to write them as

$$X_j(x) = \frac{NX_j}{DX_j}, \quad (j = 1, 2, \dots, d)$$

where $NX_j(x), DX_j(x) \in P\mathbb{C}[x_1, x_2, \dots, x_d]$ are polynomials irreducible from each other.

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Let i be one of $\{1, 2, \dots, d\}$ and denote by Σ_i the variety of zero set of DX_i . The points on Σ_i are mapped to $F(\Sigma_i)$, which is divergent. But, unless the infinity is a fixed point of the map, there is a possibility that it returns to a finite point after some steps of the map. This is the phenomenon known as singularity confinement (SC) [1]. If the points return to a finite region after m iteration of the map, we call this number m , the ‘steps of SC’. This means that none of $DX_j^{(m+1)}(\Sigma_i)$, $j = 1, 2, \dots, d$ in $F^{(m+1)}(\Sigma_i)$ is identically zero, while $F^{(m)}(\Sigma_i)$ is divergent.

It is not difficult to see how this phenomenon takes place [2]. Suppose the map has an inverse which is again rational. If Σ_- is the zero set of the denominators of the inverse map F^{-1} , it is mapped to $F^{-1}(\Sigma_-)$, which is divergent, by the inverse map. Conversely the points at infinity are mapped back to Σ_- by the forward map F . From this observation it is clear that when $F^{(m)}(\Sigma_i) \in F^{-1}(\Sigma_-)$, it is mapped to $F^{(m+1)}(\Sigma_i) \in \Sigma_-$, which is finite. This is the mechanism that the SC phenomenon undergoes.

Now we assume that the map has p invariants. It was proved in [2, 3] that, when $p \geq d/2$, periodic points of all periods form varieties of non zero dimension, provided periodic points of one period form a variety. The existence of such a variety characterizes integrability of the map. Since these varieties are specified only by the invariants, we called them invariant varieties of periodic points, or IVPP for short. We shall refer this theorem as ‘IVPP theorem’ in this paper. Moreover, in [2], we have shown, in the 3 dimensional Lotka-Volterra (3dLV) map case, that the IVPPs of all periods can be derived from SC.

Since there exist two invariants in the 3dLV map, they are sufficient to parameterize the surface Σ_i by themselves. After the recovery from the singularity, $F^{(m+1)}(\Sigma_i)$ is on Σ_- by the reason we explained above. One can repeat the map further. Notice that all such images of Σ_i are also parametrized by the invariants alone. This means that $F^{(n+1)}(\Sigma_i)$ for all $n \geq m$ are sets of rational functions of the invariants. In other words the numerators and denominators of $X_j^{(n+1)}(\Sigma_i)$ are polynomial functions of the invariants.

An important observation in [2] is that $DX_i^{(n+1)}(\Sigma_i)$ must vanish at the period n points, because $F^{(n+1)}(\Sigma_i)$ is the n -th image of $F(\Sigma_i)$, which was divergent. This is possible only if the zero set of one of the irreducible factors of the polynomial function $DX_i^{(n+1)}(\Sigma_i)$ decides the period n points. Moreover this polynomial factor of the invariants must be the one which determines the IVPP of period n , because this map has a single polynomial for each

period. In this way we could derive IVPPs of all periods simply by repeating the map.

We would like to emphasize that a direct derivation of an IVPP from the periodicity conditions by using algebraic method becomes more and more difficult as the period and/or the dimension of the map increases. On the other hand it is remarkable that our algorithm explained above enables us to derive a series of IVPPs by iteration of the map.

We can directly apply this algorithm when the number of invariants is $d - 1$. In fact it is not difficult to derive IVPP's of various maps, such as the 3d KdV map, the 4d LV map, in this method. When the number of invariants p is less than $d - 1$, a single polynomial of the invariants is not sufficient to determine IVPP of each period. Since we have not studied such cases so far, we must develop a new method. Our purpose of this paper is to solve this problem. To this end we must introduce $d - p - 1$ additional conditions by hand so that Σ_i is parameterized in terms of the invariants. One of the polynomials which determine IVPP can be derived by the same procedure of the $p = d - 1$ case. We discuss in §2 a new algorithm to determine other functions, and show in §3 how this new algorithm works in the case of 3 point Toda map with $d = 6$ and $p = 4$.

II. ALGORITHM FOR GENERAL CASES

We assume that the map (1) has p invariants,

$$H_1(x), H_2(x), \dots, H_p(x). \quad (2)$$

According to the IVPP theorem, the set of all period n points are on a variety specified by the invariants. The purpose of this section is to find a method which enables one to derive, iteratively, all IVPP's by the maps recovered from the SC.

The key point of the method used in the derivation of IVPP's of the 3dLV map was to parameterize the zero set of one of the denominators of the map in terms of the invariants [2]. We generalize this idea and propose the following new algorithm.

Algorithm

Let F be the rational map defined by (1) with p invariants (2). We assume that the step m of SC of this map is finite.

1. We choose any one of X_1, X_2, \dots, X_d , which we call X_i , and impose the conditions:

$$\left\{ DX_i(x) = 0, DX_i^{(2)}(x) = 0, \dots, DX_i^{(d-p)}(x) = 0, \right. \\ \left. H_1(x) = h_1, H_2(x) = h_2, \dots, H_p(x) = h_p \right\} \quad (3)$$

to solve for $x = (x_1, x_2, \dots, x_d)$. It enables us to parameterize Σ_i , the zero set of DX_i in terms of the values $h = (h_1, h_2, \dots, h_p)$ of the invariants. We denote by $p^{(0)}$ the points on Σ_i .

2. In the second step we iterate the map by the substitution of $p^{(0)}$ to $F^{(l)}(x)$, $l \leq m$, which must be divergent. As $d - p$ increases, the steps m of the SC will be larger.
3. By the substitution of $p^{(0)}$ to $F^{(n+1)}(x)$, $n \geq m$ we will find a single polynomial in $DX_i^{(n+1)}$ which vanishes only at the points of n period. It must be one of the independent set of functions which determine the IVPP of period n .
4. Finally, to find other elements of the IVPP set of period n , we identify the other components of $F^{(n+1)}(\Sigma_i)$ with those of $F(\Sigma_i)$, which are all written by the invariants.
5. Instead of the last step we can find the set of IVPP functions of period n as the intersection of $DX_i^{(k)} = 0$, $k = n + 1, \dots, n + d - p$.

Needless to say this algorithm becomes the one which we developed in our previous work and is applicable when $p = d - 1$.

III. 3 POINT TODA MAP

We have already derived the IVPP of period 3 of the 3 point Toda map in [3]. We would like to mention, however, that the analysis using computer algebra becomes much harder as the number of freedom of the map increases, if we derive the IVPP's directly from the periodicity conditions of the map. It was not possible to find IVPP's of periods higher than 3 by using our computer.

We apply, in this section, the new algorithm in the case of 3 point Toda map, and see how it works when p is greater than $d - 1$. We will see that we do not need to solve the periodicity conditions one by one. Our algorithm enables us to derive all IVPP's iteratively.

Map and Invariants

The 3 point Toda map is defined [5] by

$$\begin{aligned} & (X, Y, Z, U, V, W) \\ &= \left(y \frac{zu + zx + wu}{yw + yz + vw}, z \frac{xv + xy + uv}{zu + zx + wu}, x \frac{yw + yz + vw}{xv + xy + uv}, \right. \\ & \quad \left. u \frac{yw + yz + vw}{zu + zx + wu}, v \frac{zu + zx + wu}{xv + xy + uv}, w \frac{xv + xy + uv}{yw + yz + vw} \right). \end{aligned} \quad (4)$$

This map has four invariants,

$$\begin{cases} r = xyz \\ t = x + y + z + u + v + w \\ f = xy + yz + zx + uv + vw + wu + xv + yw + zu \\ g = uvw - xyz. \end{cases} \quad (5)$$

Parameterization of Σ_x and SC

According to the first step of the Algorithm, we parametrize Σ_x by using $DX^{(2)}(x)$ in (3), and obtain

$$\begin{aligned} p^{(0)} = & \left(\frac{r(-g^2t + gf^2 + f^2r)}{g^3}, \frac{g^2f}{-g^2t + f^2r}, \frac{(-g^2t + f^2r)g}{(-g^2t + gf^2 + f^2r)f}, \right. \\ & \left. -\frac{(r+g)(-g^2t + f^2r)}{g^3}, -\frac{g(-g^2t + gf^2 + f^2r)}{(-g^2t + f^2r)f}, \frac{g^2f}{-g^2t + gf^2 + f^2r} \right). \end{aligned} \quad (6)$$

We can proceed easily the second and the third steps of the Algorithm to see how the SC map undergoes

$$\begin{aligned} p^{(0)} & \rightarrow \left(\infty, -\frac{g}{f}, 0, 0, \frac{g}{f}, \infty \right) \rightarrow \left(\frac{0}{0}, 0, \frac{0}{0}, 0, \frac{0}{0}, \frac{0}{0} \right) \\ & \rightarrow \left(0, \frac{0}{0}, \frac{0}{0}, 0, \frac{0}{0}, \frac{0}{0} \right) \rightarrow \left(-\frac{g}{f}, \infty, 0, 0, \infty, \frac{g}{f} \right) \rightarrow p^{(5)} \rightarrow \dots \end{aligned} \quad (7)$$

where

$$\begin{aligned} p^{(5)} = & \left(\frac{g^2f}{-g^2t + f^2r}, \frac{r(-g^2t + gf^2 + f^2r)}{g^3}, \frac{(-g^2t + f^2r)g}{(-g^2t + gf^2 + f^2r)f}, \right. \\ & \left. -\frac{(r+g)(-g^2t + f^2r)}{g^3}, -\frac{g(-g^2t + gf^2 + f^2r)}{(-g^2t + f^2r)f}, \frac{g^2f}{-g^2t + gf^2 + f^2r} \right). \end{aligned} \quad (8)$$

Here 0/0 means that the denominator and the numerator of the component become zero separately, so that we can not determine its value. In other words the point is indeterminate.

By the fourth step of the Algorithm we find the IVPP of period 4 as the intersection of

$$-g^2t + f^2r = 0, \quad tf + g = 0.$$

We can apply the fifth step of the Algorithm to obtain the IVPP of period 3. Namely from (7) we see that $DX^{(4)} = f$ and $DX^{(1)} = 0$, from which we find one of the 3 period conditions, $f = 0$. Another period 3 condition must follow from the condition $DX^{(5)} = -g^2t + f^2 = 0$ since $DX^{(2)} = 0$. Combining them together we obtain the period 3 conditions

$$f = 0, \quad g^2t = 0.$$

IVPP's of the 3 point Toda map

We can derive the IVPPs of higher periods by the iteration of the map. They are given by intersections of two functions in the form $v^{(n)} = \{\gamma_1^{(n)} = 0, \gamma_2^{(n)} = 0\}$. We have found

$$\begin{aligned} \gamma_1^{(3)} &= f, & \gamma_2^{(3)} &= tg^2 \\ \gamma_1^{(4)} &= g^2t - f^2r, & \gamma_2^{(4)} &= tf + g \\ \gamma_1^{(5)} &= -2f^3r^2 + 2frtg^2 - f^3rg - g^4, & \gamma_2^{(5)} &= 4f^3rg^3 + f^6r^2 - g^6 \\ \gamma_1^{(6)} &= 5f^4r^2 + 5gf^4r - 6g^2f^2tr + g^2f^4 - 3g^3f^2t + 2g^4f + g^4t^2, \\ \gamma_2^{(6)} &= 4f^6 + 3t^2f^2g^2 - 4f^3g^2 + g^4 \\ &\text{etc.} \end{aligned}$$

Remarks

Since we have chosen the function $DX^{(2)}$ as the additional condition for the parametrization of Σ_x , the number of steps of the SC increased from 3 to 5. This change unables us to decide the IVPPs of period 2. However we can always impose periodicity conditions to find the IVPP of each period separately. For instance, for the period 2 case, we can manipulate the periodicity conditions directly, and find that the points on the intersection of the following three surfaces

$$\begin{aligned} xyu^3 - 3xyz(x + y - z + u)u - z(x^2 + yz)(y^2 + zx) &= 0 \\ yzv^3 - 3xyz(y + z - x + v)v - x(y^2 + zx)(z^2 + xy) &= 0 \\ xzw^3 - 3xyz(z + x - y + w)w - y(z^2 + xy)(x^2 + yz) &= 0 \end{aligned}$$

satisfy the period 2 conditions.

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